Homework 1, due 9/2

1. For each $p, q > 0$ real, and g, determine the radius and the center of the circle with equation $pz\bar{z} + gz + \overline{gz} = q$.

2. For each $c \in D(0, 1)$ define the transformation L_c by

$$L_c(z) = \frac{z - c}{1 - \bar{c}z}.$$

   Prove that L_c maps the unit disk $D(0, 1)$ onto the unit disk, and the unit circle $S^1 = \{z : |z| = 1\}$ onto the unit circle.

3. Prove that if $f : \mathbf{C} \to \mathbf{C}$ satisfies the Cauchy-Riemann equations at z, then $g : \mathbf{C} \to \mathbf{C}$ defined by $g(w) = \overline{f(\bar{w})}$ satisfies that Cauchy-Riemann equations at $w = \bar{z}$.

4. If $f : \Omega \to \mathbf{C}$ is holomorphic on a connected open set $\Omega \subset \mathbf{C}$, prove the following:

   (i) If $f'(z) = 0$ for all $z \in \Omega$, then f is constant.

   (ii) If there exists $c \in \mathbf{C}$ such that $f(z) = c \cdot \overline{f(z)}$ for every $z \in \Omega$, then f is constant.

   (iii) If $f(\Omega) \subset \mathbf{R}$, then f is constant.

5. The dilogarithm Li_2 is the series

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

   (i) Determine its radius of convergence R.

   (ii) Does the series converge on the closure $\overline{D(0, R)}$?